INTERNATIONAL JOURNAL OF ADVANCE RESEARCH, IDEAS AND INNOVATIONS IN TECHNOLOGY

IJARIIT

ISSN: 2454-132X

Impact Factor: 6.078

(Volume 11, Issue 1 - V11I1-1210)

Available online at: https://www.ijariit.com

Tangent Bundles Related to Various Properties of Differential Geometry

Anowar Hussain Sadiyal

sadieal@qu.edu.sa

Qassim University, Saudi Arabia

ABSTRACT

The objective of the present article is to investigate the lifts of a $F_\lambda(2\nu+3,2)$ -structure and determine its integrability requirements and partial integrability on the tangent bundle. Finally, the third tangent bundle T_3M is examined in order to study the extension of a $F_\lambda(2\nu+3,2)$ -structure.

Keywords: *Lifts, Nijenhuis Tensor, Partial Differential Equations, Projection Tensors, Integrability*

1. INTRODUCTION

The polynomial structure $Q(F)=F^n+a_nF^{n-1}+.......+a_2F+a_1I$, was described by Goldenberg et al. [9, 11], where F is a (1,1) tensor field and I is a unit tensor field on a differentiable manifold M. Gok et al. [10] recently defined and established the geometric characteristics of a $f(a,b)(3,2,1)$ -structure, $a,b\in R$ and $b\neq 0$ on M. They looked at its integrability criteria and partial integrability. Several f -structures were examined by numerous researchers in [8, 18, 19, 20, 22, 23, 30, 31, 17]. A manifold M has a tangent bundle TM. Let's examine this bundle instead. The main tool used in differential geometry to study geometrical structures and their characteristics, including integrability, curvature, Lie derivative, etc., is the tangent bundle. Structures such as almost complex structures with some fundamental features generated in tangent bundles were presented and investigated by Yano and Ishihara [29]. These lifts of an almost product structure over an almost r-contact structure, together with TM, have been studied by Das and Khan [3]. For example, Dida et al. [5, 6, 7], Khan and De [12, 16], Khan [21, 13, 27], Omran at el [24], and Peyghan et al. [25] have all contributed extensively to our understanding of geometric structures and relationships. This paper's primary goal is to determine the partial integrability and integrability requirements of an $F_\lambda(2\nu+3,2)$ -structure by studying its lifts manifolds on the tangent bundle. Finally, the third tangent bundle T_3M is examined in relation to the prolongation of a $F_\lambda(2\nu+3,2)$ -structure.

Let M ($\dim = n$) be a differentiable manifold. A tensor field $F\neq 0$ of type (1, 1) on M is said to be an $F_\lambda(2\nu+3,2)$ -structure on M if

$$F^{2\nu+3}+\lambda^r F^2=0. \tag{1.1}$$

Thus, (M,F) is called an $F_\lambda(2\nu+3,2)$ -manifold.

Let us define two l and m as projection tensors

$$l=-\frac{F^{2\nu+1}}{\lambda^r} \quad \text{and} \quad m=I+\frac{F^{2\nu+1}}{\lambda^r} \tag{1.2}$$

From (1.2), we obtained

$$l+m=0,$$
$$l^2=l,\quad m^2=m,\quad lm=ml=0,$$
$$Fl=lF=F,\quad Fm=mF=0,$$

Let D_l and D_m be two complemetary distributions in M corresponding l and m and rank of F is r.

2. THE COMPLETE LIFT OF AN $F_\lambda(2\nu+3,2)$ -STRUCTURE IN TM

$TM=\cup_{p\in M}T_pM$ is a tangent bundle over the manifold M, assuming that M is a n-dimensional differentiable manifold of class C^∞ and that T_pM is the tangent space at a point p of M. A differentiable manifold of length 2n is the tangent bundle TM of M. In M, let $\Im_b^a$ represent the set of tensor fields of class C∞ and type (a,b), and $\Im_b^a(T(M))$ represent the equivalent set of tensor fields in the tangent bundle of M with dimension 2n is represented by TM. The set of tensor fields in M and TM are represented by $\Im_b^a(M)$ and $\Im_b^a(T(M))$, respectively.

Let $\forall F,G\in\Im_1^1(M)$. Then [28]

$$(FG)^C=F^CG^C. \tag{2.1}$$

Set $G=F$ in above equation, we obtained

$$(F^2)^C=(F^C)^2. \tag{2.2}$$

Also,

$$(F+G)^C=F^C+G^C. \tag{2.3}$$

Employing the complete lift on (1.1), we obtained

$$(F^{2\nu+3}+\lambda^rF^2)^C=0.$$

In the view of (2.2), we acquire

$$(F^C)^{2\nu+3}+\lambda^r(F^C)^2=0. \tag{2.4}$$

Thus from (1.1), (2.4) and [28], the rank (F^C) = 2r iff the rank (F) = r. Hence, we obtained

Theorem 2.1 *Let $F\in\Im_1^1(M)$ be an $F_\lambda(2\nu+3,2)$-structure, then F^C is also an $F_\lambda(2\nu+3,2)$-structure in TM.*

Theorem 2.2 *The $F_\lambda(2\nu+3,2)$-structure F of rank r in M iff F^C is of rank r in TM.*

3. INTEGRABILITY OF $F_\lambda(2\nu+3,2)$ -*STRUCTURE IN* TM.

Let $N\in\Im_1^2(M)$ be the Nijenhuis tensor of F and defined by [3, 14, 15]

$$N(U,V)=[FU,FV]-F[FU,V]-F[U,FV]+F^2[U,V]. \tag{3.1}$$

Employing the complete lifts on the above equation, we obtained

$$\begin{aligned} N^C(U^C,V^C)=&[F^CU^C,F^CV^C]-F^C[F^CU^C,V^C]\\ &-F^C[U^C,F^CV^C]+(F^2)^C[U^C,V^C], \end{aligned} \tag{3.2}$$

where $N^C\in\Im_1^2(TM)$ is the Nijenhuis tensor of F^C.

Let $\forall U,V\in\Im_0^1(M)$ and $F\in\Im_1^1(M)$, we get

$$\begin{aligned} &[U^C,V^C]=[U,V]^C,\\ &(U^C+V^C)=U^C+V^C,\\ &F^CU^C=(FU)^C. \end{aligned} \tag{3.3}$$

From (1.3) and (3.3), we obtained

$$\begin{aligned} &F^Cl^C=(Fl)^C=F^C,\\ &F^Cm^C=(Fm)^C=0. \end{aligned} \tag{3.4}$$

Theorem 3.1 *Let $\forall U,V\in\Im_0^1(M)$, then*

$$N^C(m^CU^C, m^CV^C) = l[m^CU^C, m^CV^C], \quad (3.5)$$
$$m^CN^C(U^C, V^C) = m^C[F^CU^C, F^CV^C], \quad (3.6)$$
$$m^CN(l^CU^C, l^CV^C) = m^C[F^CU^C, F^CV^C], \quad (3.7)$$
$$m^CN^C((F^2)^CU^C, (F^2)^CV^C) = m^CN^C[l^CU^C, l^CV^C], \quad (3.8)$$

Proof: The required results follow from (1.3), (3.4) and (3.1).

Theorem 3.2 *Let* $\forall U,V \in \Im_0^1(M)$*, the given criteria are equivalent*

(i) $m^CN^C(U^C, V^C) = 0,$

(ii) $m^CN(l^CU^C, l^CV^C) = 0,$

(iii) $m^CN^C((F^2)^CU^C, (F^2)^CV^C) = 0.$

Proof: From (3.8), we obtained

$$N^C(l^CU^C, l^CV^C) = 0 \leftrightarrow N^C((F^2)^CU^C, (F^2)^CV^C) = 0. \quad (3.9)$$

In view of (3.6), (3.7) and (3.9), we obtained that criteria (i), (ii), and (ii) are equivalent.

Theorem 3.3 *Let* D_m^C *be the complete lift of* D_m *in* TM*. Then* D_m^C *is integral if* D_m *is integrable in* M*.*

Proof: Let $\forall U,V \in \Im_0^1(M)$*. Then* D_m *is integrable iff [28]*

$$l[mU, mV] = 0, \quad (3.10)$$

where $l = I - m.$

Taking complete lift of (3.10), we acquire

$$l^C[m^CU^C, m^CV^C] = 0, \quad (3.11)$$

where $l^C = (I-m)^C = I - m^C$*. Hence, this completes the proof.*

Theorem 3.4 *Let* D_m^C *be the complete lift of* D_m *in* TM*. Then* D_m^C *in* TM *is integral if* $N^C(m^CU^C, m^CV^C) = 0, \forall U,V \in \Im_0^1(M).$

Proof: Let $\forall U,V \in \Im_0^1(M)$*. Then* D_m *is integrable iff [28]*

$$N[mU, mV] = 0.$$

From (3.5), we obtained

$$N^C(m^CU^C, m^CV^C) = (F^2)^C[m^CU^C, m^CV^C]. \quad (3.12)$$

Operating l^C *on (3.12), we get*

$$l^CN^C(m^CU^C, m^CV^C) = (F^2)^Cl^C[m^CU^C, m^CV^C]. \quad (3.13)$$

Using (3.11), the equation (3.13) reduces to

$$l^CN^C(m^CU^C, m^CV^C) = 0. \quad (3.14)$$

Also, we have

$$l^CN^C(m^CU^C, m^CV^C) = 0. \quad (3.53)$$

Combining (3.14) and (3.15), we obtained

$$(l^C + m^C)N^C(m^CU^C, m^CV^C) = 0.$$

Since $l^C + m^C = I$*, we obtained*

$$N^C(m^CU^C, m^CV^C) = 0.$$

Theorem 3.5 *Let* $\forall U,V \in \Im_0^1(M)$ *and* D_l *be integrable in* M *such that* $mN(U,V) = 0$*. Then* D_l^C *is integrable in* TM *iff any condition of Theorem (3.2) is satisfied.*

Proof: If D_l *be integrable in* M *iff*

$$mN(lU,lV)=0.$$

Thus D_l^C *be integrable in* TM *iff*

$$m^C N^C(l^C U^C, l^C V^C)=0.$$

From (3.8), we obtained the required results.

Theorem 3.6 *Let* F^C *be the complete lift of an* $F_\lambda(2\nu+3,2)$ *-structure* F *in* TM. *Then* F^C *is partially integrable in* TM *iff F is partially integrable in M.*

Proof: Let $\forall U,V\in\Im_0^1(M)$ *and an* $F_\lambda(2\nu+3,2)$ *-structure F in M is partially integrable iff*

$$N(lU,lV)=0. \qquad (3.16)$$

From (1.3) and (3.1), we acquire

$$N^C(l^C U^C, l^C V^C)=(N(lU,lV))^C,$$

which implies

$$N^C(l^C U^C, l^C V^C)=0 \leftrightarrow N(lU,lV)=0.$$

Also from Theorem (3.2), $N^C(l^C U^C, l^C V^C)=0$ *is equivalent to*

$$N^C((F^2)^C U^C, (F^2)^C V^C)=0.$$

This completes the proof.

Theorem 3.6 *Let* F^C *be the complete lift of an* $F_\lambda(2\nu+3,2)$ *-structure* F *in* TM. *Then* F^C *is partially integrable in* TM *iff F is partially integrable in M.*

Proof: Let $\forall U,V\in\Im_0^1(M)$ *and an* $F_\lambda(2\nu+3,2)$ *-structure F in M is partially integrable iff*

$$N(U,V)=0. \qquad (3.17)$$

Taking the complete lift on (3.1), we obtained

$$N^C(U^C,V^C)=(N(U,V))^C.$$

From $N(U,V)=0$, *then we obtained*

$$N^C(U^C,V^C)=0.$$

This completes the proof.

4. THE HORIZONTAL LIFT OF AN $F_\lambda(2\nu+3,2)$ -STRUCTURE IN TM

Let S *and* $\nabla_\lambda S$ *be tensor fields in M and T M, respectively and given by in the term of partial differential equations [1, 2, 28]*

$$S=S_{k\ldots j}^{i\ldots h}\frac{\partial}{\partial U^i}\otimes\ldots\ldots\otimes\frac{\partial}{\partial U^h}\otimes dx^k\otimes\ldots\ldots\otimes dx^j,$$

$$\nabla_\lambda S=V^l\nabla_\lambda S_{k\ldots j}^{i\ldots h}\frac{\partial}{\partial U^i}\otimes\ldots\ldots\otimes\frac{\partial}{\partial V^h}\otimes dx^k\otimes\ldots\ldots\otimes dx^j,$$

where ∇ *is an affine connection, corresponding to the induced coordinates* (U^h,V^h) *in* $\pi^{-1}(U)$ *[28].*

The horizontal lift of S *is denoted by* S^H *in T M and given by*

$$S^H=S^C-\nabla_\lambda S.$$

Theorem 4.1 *Let* $F\in\Im_1^1(M)$ *be an* $F_\lambda(2\nu+3,2)$ *-structure, then* F^H *is also an* $F_\lambda(2\nu+3,2)$ *-structure in* TM.

Proof: Let F^H *be the horizontal lift of* F *in* TM. *Taking the horizontal lift on (1.1), we obtained*

$$(F^{2\nu+3}+\lambda^r F^2)^H=0.$$

Since $(F^2)^H=(F^H)^2$, *then*

$$(F^H)^{2\nu+3}+\lambda^r(F^H)^2=0. \qquad (4.1)$$

Hence, F^H *is also an* $F_\lambda(2\nu+3,2)$ *-structure in* TM.

Theorem 4.2 *The* $F_\lambda(2\nu+3,2)$ *-structure* F *of rank* r *in M iff* F^H *is of rank* $2r$ *in* TM.

Proof: In the view of from (1.1), (4.1) and [28], the rank (F^H) *= 2r iff the rank* (F) *= r.*

If $m \in \Im_0^1(M)$ *be a projection tensor field such that*

$$m^2 = m.$$

Taking the horizontal lift and from [28], we obtained

$$(m^H)^2 = m^H.$$

This shows that m^H *is a projection in T M. Then* D^H *is the distribution in T M corresponding to* m^H, *where D is the distribution in M.*

5. PROLONGATION OF AN $F_\lambda(2\nu+3,2)$ -STRUCTURE IN T_3M

Let T_3M *be the third order tangent bundle in M and let* F^{III} *be the third lift on F in* T_3M. $\forall F, G \in \Im_1^1(M)$, *we obtained*

$$\begin{aligned}(G^{III}F^{III})U^{III} &= G^{III}(F^{III}U^{III})\\ &= (G^{III}(FU)^{III})\\ &= (G(FU))^{III}\\ &= (GF)^{III}U^{III},\end{aligned}$$

$\forall U \Im_0^1(M)$. *Thus we get*

$$G^{III}F^{III} = (GF)^{III}.$$

Theorem 5.1 *Let* $F \in \Im_1^1(M)$ *be an* $F_\lambda(2\nu+3,2)$ *-structure, then third lift* F^{III} *is also an* $F_\lambda(2\nu+3,2)$ *-structure in* TM.

Proof: Let F^{III} *be the horizontal lift of* F *in* TM. *Taking the horizontal lift on (1.1), we obtained*

$$(F^{2\nu+3} + \lambda^r F^2)^{III} = 0.$$

Since $(F^2)^{III} = (F^{III})^2$, *then*

$$(F^{III})^{2\nu+3} + \lambda^r (F^{III})^2 = 0.$$

Hence, F^{III} *is also an* $F_\lambda(2\nu+3,2)$ *-structure in* TM.

Theorem 5.2 *Let* $F \in \Im_1^1(M)$ *be an* $F_\lambda(2\nu+3,2)$ *-structure such that* F^{III} *is integrable in* T_3M *iff F is integrable in M.*

Proof: Let N^{III} *and* N be Nijenhuis tensors of F^{III} and F. Then we have

$$N^{III}(U,V) = (N(U,V))^{III}. \qquad (3.17)$$

As an $F_\lambda(2\nu+3,2)$ *-structure is integrable in M iff N (U, V) = 0, then from above equation, we obtained*

$$N^{III}(U,V) = 0.$$

Hence F^{III} *is integrable in* T_3M *iff F is integrable in M.*

REFERENCES

[1] M. Altunbas, L. Bilen and A. Gezer, Remarks about the Kaluza-Klein metric on tangent bundle, Int. J. Geo. Met. Mod. Phys., 16 (3) (2019), 1950040.
[2] S. Azami, General natural metallic structure on tangent bundle, Iran J Sci Technol Trans Sci 42(2018), 81-88.
[3] L. S. Das and M. N. I. Khan, Almost r-contact structure in the tangent bundle, Differential Geometry-Dynamical System, 7 (2005), 34-41.
[4] L. S. Das, R. Nivas and M. N. I. Khan, On submanifolds of codimension 2 immersed in a hsu–quarternion manifold, Acta Mathematica Academiae Paedagogicae Ny´ıregyh´aziensis 25 (1), 129-135, 2009.
[5] H. M. Dida and F. Hathout, Ricci soliton on the tangent bundle with semisymmetric metric connection, Bulletin of the Transilvania University of Bra¸sov Series III: Mathematics and Computer Science, 1(63)(2) (2021), 37-52.

[6] H. M. Dida and A. Ikemakhen, A class of metrics on tangent bundles of pseudo-Riemannian manifolds, Archivum Mathematicum (BRNO) Tomus, 47 (2011), 293–308.
[7] H. M. Dida, F. Hathout and M. Djaa, On the geometry of the second order tangent bundle with the diagonal lift metric, Int. Journal of Math. Analysis, 3(9) 2009, 443 - 456.
[8] S. I. Goldberg, K. Yano, Polynomial structures on manifolds, Kodai Math Sem Rep., 22 (1970) 199-218.
[9] S. I. Goldberg and K. Yano, Polynomial structures on manifolds, Kodai Math Sem Rep., 22 (1970), 199-218.
[10] M. G¨ok, E. Kili¸c and C. Ozg¨ur, ¨ f(a, b)(3, 2, 1)-structures on manifolds, Journal of Geometry and Physics, 169 (2021), 104346.
[11] S. I. Goldberg and N. C. Petridis, Differentiable solutions of algebraic equations on manifolds, Kodai Math. Sem. Rep., 25 (1973), 111–128.
[12] M. N. I. Khan and U. C. De, Liftings of metallic structures to tangent bundles of order r, AIMS Mathematics, 7(5), (2022), 7888-7897.
[13] M. N. I. Khan, Proposed theorems for lifts of the extended almost complex structures on the complex manifold, Asian-European Journal of Mathematics, 15(11) (2023), 2250200.
[14] M. N. I. Khan, Integrability of the Metallic Structures on the Frame Bundle, Kyungpook Mathematical Journal 61 (4) 2021, 791-803.
[15] M. N. I. Khan, Novel theorems for metallic structures on the frame bundle of the second order, Filomat, 36(13) (2022), 4471-4482.
[16] M. N. I. Khan and U. C. De, Lifts of metallic structure on a cross-section, Filomat, 36(18), 6369-6373, 2022.
[17] S. Kobayashi and K. Nomizu, Foundations of differential geometry, 1, Interscience, New York, 1963.
[18] H. Nakagawa, f-structures induced on submanifolds in spaces, almost Hermitian or Kaehlerian, Kodai Math. Semin. Rep., 18(2)(1966), 161–183.
[19] K. Niedzialomski, On the frame bundle adapted to a submanifold, Math. Nachr., 288(5–6)2015, 648–664.
[20] A. Naveira, A classification of Riemannian almost-product manifolds. Rend Di Mat Di Roma, 3(1983) 577-592.
[21] M. N. I. Khan, U. C. De and L. S. Velimirovic, Lifts of a quartersymmetric metric connection from a Sasakian manifold to its tangent bundle. Mathematics, 11 (2023), 53.
[22] T. Okubo, On the differential geometry of frame bundles, Ann. Mat, Pura Appl., 72(1966), 29-44.
[23] M. Ozkan, Prolongations of golden structures to tangent bundles, Differential Geometry Dynamical Systems, 16(2014), 227-239.
[24] T. Omran, A. Sharffuddin, S. I. Husain, Lift of structures on manifold, Publications De L'institut Mathematique (Beograd) (N.S.), 36(50) (1984), 93-97.
[25] E. Peyghan, F. Firuzi and U. C. De, Golden Riemannian structures on the tangent bundle with g-natural metrics, Filomat, 33 (8) (2019), 2543-2554.
[26] A. Sardar, M. N. I. Khan and U. C. De, EtaRicci Solitons and Almost co-K¨ahler Manifolds, Mathematics 9 (24) (2021), 3200.
[27] M. N. I. Khan, A. Haseeb, F-structures manifolds, Results in Nonlinear Analysis, 7(1), 8-13, 2024.
[28] K. Yano and S. Ishihara, Tangent and Cotangent Bundles. Marcel Dekker Inc., New York, 1973.
[29] K. Yano and S. Ishihara, Almost complex structures induced in tangent bundles, Kodai Math. Sem. Rep., 19 (1967), 1-27.
[30] K. Yano, C. S. Houh and B. Y. Chen, Structure defined by a tensor field phi of type (1,1) satisfying $\phi^4 \pm \phi^2 = 0$, Tensor, N. S., 23(1972), 81-87.
[31] K. Yano, On a structure defined by a tensor field f of type (1, 1) satisfying $f^3 + f = 0$, = 0, Tensor, N. S., 14(1963), 99–109